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LETTER TO THE EDITOR

Warning on multi-soliton solutions to Zakharov equations

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Abstract. Although it has been usually recognised that the Zakharov equations in one dimension are not completely integrable, multi-soliton solutions to them have been published recently, where the direct methods developed by Hirota are used. It is shown that this result is incorrect.

One of the most fundamental problems in plasma physics, and currently an area of active research, is the one of strongly turbulent plasmas. In an unmagnetised plasma, where Langmuir waves predominate, if the temperature of ions is negligible in comparison with that of electrons, high frequency electrostatic oscillations produce a ponderomotive force that depletes the ion density. Such depression plays the role of a well that traps selfconsistently the high frequency oscillations. This phenomenon was studied by Zakharov (1972) in a seminal work, where the equations that describe it were derived. Recent work on this subject has been reviewed by Rudakov and Tsytovich (1978) and Thornhill and ter Haar (1978).

The Zakharov equations in one spatial dimension, generally speaking, describe the evolution of an almost monochromatic pump wave in an homogeneous medium which is strongly dispersive, weakly nonlinear, and whose response time is finite. In dimensionless units they may be written as

$$\mathrm{i}E_t + E_{xx} - \rho E = 0, \tag{1}$$

$$\rho_{tt} - \rho_{xx} = |E|^2_{xx}, \tag{2}$$

where E is the electric field, and ρ is the ion density in units of the uniform unperturbed density n_0 (Thornhill and ter Haar 1978).

These equations are known to have a one-soliton solution, but in contrast with completely integrable systems, such as the KdV equation or the nonlinear Schrödinger equation, which have an infinite number of conserved quantities, they only have three, which may be derived from a Lagrangian (Gibbons *et al* 1977).

In addition, there are numerical works that show that when two solitary waves interact, they do not simply pass through each other, as would be the case if the equations were completely integrable. Instead, they are slowed down as ion sound is generated, and under certain circumstances they may merge (Degtyarev *et al* 1975, Payne *et al* 1982).

In contrast with this, multi-soliton solutions to Zakharov equations have been published, obtained by Hirota's technique (Ma 1979). Similar calculations are made for other systems of equations. This result is incorrect, and the purpose of this work is to show the reason.

Ma (1979) follows closely the steps of Hirota who solved, among others, the nonlinear Schrödinger equation,

$$\mathrm{i}E_t + E_{xx} + |E|^2 E = 0, \tag{3}$$

through a relatively simple method, which we shall describe briefly (Hirota 1973, 1980).

Let us take

$$E(x, t) = G(x, t)/F(x, t), \tag{4}$$

where G and F are complex and real functions respectively. Then, the Zakharov equations may be rewritten as the following system of bilinear differential equations:

$$\mathrm{i}D_t G \cdot F + D_x^2 G \cdot F = 0, \tag{5}$$

$$-D_t^2 F \cdot F + D_x^2 F \cdot F = GG^*, \tag{6}$$

where we have used the definition

$$D_h^n(k \cdot k') \equiv (\partial/\partial h - \partial/\partial h')^n k(h) k'(h')|_{h=h'} \tag{7}$$

and

$$\rho = -2[\ln F(x, t)]_{xx}. \tag{8}$$

It should be noticed that the only difference between the case of the Zakharov equations and that of the nonlinear Schrödinger equation is that the first term in (6) does not appear for the latter.

Solutions for (5) and (6) may be attempted by expanding F and G in terms of a parameter ε:

$$F = 1 \quad \varepsilon^2 f_2 + \varepsilon^4 f_4 + \ldots, \tag{9}$$

$$G = \varepsilon g_1 + \varepsilon^3 g_3 + \ldots. \tag{10}$$

Then, the following hierarchy of equations arises:

$$\mathrm{i}\partial g_1/\partial t + \partial^2 g_1/\partial x^2 = 0, \tag{11}$$

$$2(-\partial^2 f_2/\partial t^2 + \partial^2 f_2/\partial x^2) = |g_1|^2, \tag{12}$$

$$\mathrm{i}\, \partial g_3/\partial t + \partial^2 g_3/\partial x^2 = -[\mathrm{i}D_t + D_x^2](g_1 \cdot f_2) \tag{13}$$

$$2(-\partial^2 f/\partial t^2 + \partial^2 f_4/\partial x^2) = g_1 g_3^* + g_3 g_1^* - [-D_t^2 + D_x^2](f_2 \cdot f_2), \tag{14}$$

and so on. Thus, the idea is to solve (11) for g_1, then (12) for f_2, etc. One would get an exact solution if it were possible to terminate the series.

Equation (11) may be solved as a sum of exponentials

$$g_1 = \sum_{k=1}^{N} \exp[\hat{\eta}_k(x, t)], \tag{15}$$

with $\hat{\eta}_k(x, t) = P_k x - \Omega_k t - \eta_k^0$, where P_k and Ω_k are complex values that satisfy

$$-\mathrm{i}\Omega_k + P_k^2 = 0, \tag{16}$$

and η_k^0 are complex constants that depend on initial values.

When $N=1$, then $f_2 = a\exp(\hat{\eta}_1+\hat{\eta}_1^*)$, where $a=\{2[(P_1+P_1^*)^2-(\Omega_1+\Omega_1^*)^2]\}^{-1}$, and the right-hand side of (13) is

$$-[\mathrm{i}D_t+D_x^2](g_1\cdot f_2) = -(\mathrm{i}\Omega_1^*+P_1^*)a\exp(\hat{\eta}_1+\hat{\eta}_1^*)=0, \tag{17}$$

because of (16). Thus, the series terminates, so that

$$F=1+a\exp(\hat{\eta}_1+\hat{\eta}_1^*), \qquad G=\exp(\hat{\eta}_1), \tag{18}$$

where ε is absorbed by $\hat{\eta}_1$, and if we take $P_1=\eta_1+\mathrm{i}\xi$, where η_1 and ξ_1 are real, then we get the one-soliton solution

$$E(x,t)=\sqrt{2}\eta_1(1-4\xi_1^2)^{1/2}\operatorname{sech}[\eta_1(x-x_0)-2\xi_1\eta_1 t]$$
$$\times\exp[\mathrm{i}\xi_1 x+\mathrm{i}(\eta_1^2-\xi_1^2)t-\eta_{1\mathrm{I}}^0] \tag{19}$$

and

$$\rho(x,t)=-2\eta_1^2\operatorname{sech}^2[\eta_1(x-x_0)-2\xi_1\eta_1 t] \tag{20}$$

where

$$x_0=(1/2\eta_1)\ln(8\eta_1^2-32\eta_1^2\xi_1^2)+\eta_{1\mathrm{R}}^0/\eta_1. \tag{21}$$

Then it would be possible to find a two-soliton solution if the series terminated in a similar way for $N=2$. Now

$$f_2=a(1,1^*)\exp(\hat{\eta}_1+\hat{\eta}_1^*)+a(1,2^*)\exp(\hat{\eta}_1+\hat{\eta}_2^*)$$
$$+a(2,1^*)\exp(\hat{\eta}_2+\hat{\eta}_1^*)+a(2,2^*)\exp(\hat{\eta}_2+\hat{\eta}_2^*), \tag{22}$$

where

$$a(k,l^*)=\{2[(P_k+P_l^*)^2-(\Omega_k+\Omega_l^*)^2]\}^{-1}, \tag{23}$$

and the right-hand side of (13) is

$$-[\mathrm{i}D_t+D_x^2](g_1\cdot f_2)$$
$$=\{a(1,1^*)[-\mathrm{i}(-\Omega_2+\Omega_1+\Omega_1^*)-(P_2-P_1-P_1^*)^2]$$
$$+a(2,1^*)[-\mathrm{i}(-\Omega_1+\Omega_2+\Omega_1^*)-(P_1-P_2-P_1^*)^2]\}\exp(\hat{\eta}_1+\hat{\eta}_2+\hat{\eta}_1^*)$$
$$+\{a(2,2^*)[-\mathrm{i}(-\Omega_1+\Omega_2+\Omega_2^*)-(P_1-P_2-P_2^*)^2]$$
$$+a(1,2^*)[-\mathrm{i}(-\Omega_2+\Omega_1+\Omega_2^*)-(P_2-P_1-P_2^*)^2]\}\exp(\hat{\eta}_1+\hat{\eta}_2+\hat{\eta}_2^*). \tag{24}$$

According to Ma (1979),

$$g_3=a(1,2)a(1,1^*)a(2,1^*)\exp(\hat{\eta}_1+\hat{\eta}_2+\hat{\eta}_1^*)$$
$$+a(1,2)a(1,2^*)a(2,2^*)\exp(\hat{\eta}_1+\hat{\eta}_2+\hat{\eta}_2^*), \tag{25}$$

but when substituting this form of g_3 in the left-hand side of (13), there is a difference with the right-hand side, (14), that may be written as

$$\Delta=\frac{(P_1+P_1^*)(P_2+P_1^*)(P_1-P_2)[P_2(P_1^{*2}-P_1^2)+P_1(P_2^2-P_1^{*2})+P_1^*(P_2^2-P_1^2)]}{[(P_1+P_1^*)^2+(P_1^2-P_1^{*2})^2][(P_2+P_1^*)^2+(P_2^2-P_1^{*2})^2]}. \tag{26}$$

In fact, if the procedure were followed properly, it would be found that the series F and G could not be terminated in this case. It is a different matter for the case of

(3), where the first term in (6) does not exist, and as a consequence, the term in the square brackets in (26) is zero.

Thus, Hirota's technique allows to find the one-soliton solution for the Zakharov equations in a rather simple way, but two, and more than two soliton solutions cannot be found by induction.

If one looks at the asymptotic limits, when $|t| \to \infty$, of the expressions proposed by Ma (1979) as solutions, there seems to be nothing wrong with them, since the error enters only as a change in the position of each soliton, and does not introduce a major change in the forms and speeds of the solitary waves. When the two solitons get close to each other, however, the error becomes evident, since there is no conservation of the quantities that should be conserved.

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